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H_∞ OPTIMAL CONTROL THEORY OVER A FINITE HORIZON

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In this report a finite horizon H_∞ optimal control problem is treated. Results guaranteeing the existence of a unique optimal control and the worst exogenous input are derived. A criterion for the evaluation of the infimal H_∞ norm is then given in terms of the least positive value for which a certain boundary value problem has a nontrivial solution. Once the infimal value is known, a noninfimal value can be selected and suboptimal H_∞ controllers can be synthesized. The problem of synthesizing suboptimal H_∞ controllers is also considered in a very general case. Without making use of any transformations, expressions for the output feedback controller are derived in terms of solutions of two dynamic Riccati equations. In the time-invariant case, the solutions of these equations usually converge to constant matrices.

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CONTENTS

1. INTRODUCTION

There is a current need for a robust multivariable flight control design technique which gives good performance under changing environment, disturbances, and uncertainty in modeling. Also, it is very useful to know the maximum possible performance under worst case conditions. Design techniques based on H_∞ methods are ideally suited for yielding good performance of the aircraft even under worst case conditions. Thus, at the Naval Air Development Center, efforts are underway to demonstrate the feasibility and advantages of flight control design techniques based on finite horizon H_∞ techniques.

The H_∞ optimal control problem has received considerable attention recently and it is well-known that H_∞ suboptimal controllers can be synthesized via the solution of two algebraic Riccati equations under certain restrictive assumptions[1]. These assumptions can be removed using various transformations[2], and thus controller synthesis can be accomplished in the general case in an indirect manner. Recently the techniques have been extended to time-varying systems[3] under similar restrictive assumptions. One of the contributions of the present report is the derivation of the results in a general case in the time-varying setting. The synthesis of the controller is accomplished for a general error criterion in the finite horizon case and the implementation of the equations on a digital computer is easy. The synthesis can be accomplished by means of two dynamic Riccati equations, one related to the controller part and the other to the observer part. In the time-invariant case, the solutions of these equations usually tend to constant matrices.

A lot of current research has also been directed towards the problem of estimating the infimal H_∞ norm of a given system. There are a few iterative techniques in the time-invariant case. There are virtually no techniques in the time-varying case and another contribution of this report is an efficient technique for the estimation of the infimal H_∞ norm in a very general setting. The technique consists of considering the inherent minimax problem and treating the adjoint variables associated with the maximization problem as state variables for the minimization problem. Once this is accomplished, the techniques of [4-9] can be applied to get the infimal norm.

We treat the problem of existence and computation of the minimal H_∞ norm initially. Then the problem of synthesizing the suboptimal H_∞ controllers will be taken up. In the derivation of the output feedback controller, the full state feedback expressions are useful in the definition of the gain of the controller part and duality plays an important role in the assignment of the gain of the observer.

In all control problems it is very useful to know the maximum possible performance under worst case conditions. The theory of Sections 2-6 is useful in obtaining a quantitative idea of achievable performance. Since suboptimal design is more practical, this topic will be treated in Sections 7-10. Also the command following problem can be suitably recast to fit the problem formulation of Section 7.

In the time-invariant case, the solutions of the dynamic Riccati equations involved usually tend towards constant matrices, if the final time is large enough. This is very

valuable in the case of flight control design. Research is also being done to successfully deal with parameter variations under the present setting. In our reserach we are mainly interested in the robust performance aspects of the aircraft under variations of the system model, as opposed to robust stability considerations.

2. Existence of Optimal Inputs

In this section we obtain results for the existence of optimal exogenous and control inputs for the finite horizon H_∞ problem. Consider the system given by

$$\dot{x} = A(t)x + B_1(t)u + B_2(t)v, \quad x(0) = 0. \tag{2.1}$$

$$z = C(t)x + D(t)u + E(t)v, \tag{2.2}$$

where x and z are the state vector and the error vector respectively. The matrices $A(t), B_1(t), B_2(t), C(t), D(t)$, and $E(t)$ will be assumed to be continuous on $[0, T]$, where T is the final time. In addition $u, v \in L_2(0, T)$.

The problem is to show the existence of u and v for which

$$\inf_{v \neq 0} \sup_{u} \frac{\int_0^T \frac{1}{2} v^* R v \, dt}{\int_0^T \frac{1}{2} z^* W z \, dt} \tag{2.3}$$

is achieved. In (2.3) $R(t)$ and $W(t)$ are continuous positive definite matrices on $[0, T]$.

We now consider the maximization part in (2.3). We will show that given any v, there exists a u which minimizes $\int_0^T z^* W z \, dt$. By simple changes in variables, the above minimization problem can be converted to a problem of the form considered in Example 2, Section 3.3 of [10]. From the results of [10], there exists a unique u which minimizes $\int_0^T z^* W z \, dt$.

We can write the functional in (2.3) as

$$J(u, v) = \frac{\int_0^T \frac{1}{2} v^*(t) R(t) v(t) \, dt}{\int_0^T \{\frac{1}{2} x^* W_1 x + x^* W_2 u + \frac{1}{2} u^* W_3 u + x^* W_4 v + \frac{1}{2} v^* W_5 v + u^* W_6 v\} \, dt}. \tag{2.4}$$

We will use the adjoint variables associated with the maximization part of (2.3) as state variables for the minimization part of (2.3).

Let

$$\lambda = \inf_{v \neq 0} \sup_{u} J(u, v). \tag{2.5}$$

Since we established the existence of a maximizing u for any given $v \neq 0$, it only remains to establish the existence of a minimizing v to guarantee the existence of λ.

Let θ be the adjoint variable associated with the maximization problem. For any $v \neq 0$, we need to select u to minimize

$$\int_0^T \{\frac{1}{2}x^* W_1 x + x^* W_2 u + \frac{1}{2}u^* W_3 u + x^* W_4 v + \frac{1}{2}v^* W_5 v + u^* W_6 v\}\, dt \qquad (2.6)$$

From the maximum principle[10], which in this case is also a sufficient condition for optimality because of the uniqueness of the optimal u, the Hamiltonian is given by

$$H = -\{\frac{1}{2}x^* W_1 x + x^* W_2 u + \frac{1}{2}u^* W_3 u + x^* W_4 v + \frac{1}{2}v^* W_5 v + u^* W_6 v\} + \theta^* \{A(t)x + B_1(t)u + B_2(t)v\}. \qquad (2.7)$$

The adjoint variable θ satisfies

$$\frac{d\theta}{dt} = W_1 x + W_2 u + W_4 v - A^* \theta, \qquad (2.8)$$

with

$$x(0) = 0, \;\; \theta(T) = 0. \qquad (2.9)$$

Assuming that W_3 is invertible for all $t \in [0, T]$, the optimal controller is given by

$$u = W_3^{-1}(B_1^* \theta - W_2^* x - W_6 v). \qquad (2.10)$$

Let

$$\begin{aligned} \hat{A} &= A - B_1 W_3^{-1} W_2^*, \\ \hat{B} &= B_1 W_3^{-1} B_1^*, \\ \hat{C} &= W_1 - W_2 W_3^{-1} W_2^*, \\ G_1 &= B_2 - B_1 W_3^{-1} W_6, \\ G_2 &= W_4 - W_2 W_3^{-1} W_6. \end{aligned} \qquad (2.11)$$

Thus we have

$$\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & -\hat{A}^* \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} v, \qquad (2.12)$$

with

$$x(0) = 0, \;\; \theta(T) = 0. \qquad (2.13)$$

Let

$$\zeta = \begin{pmatrix} x \\ \theta \end{pmatrix}, \tag{2.14}$$

$$M = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & -\hat{A}^* \end{pmatrix}, \tag{2.15}$$

$$N = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \tag{2.16}$$

$$U_z = W_3^{-1} (-W_2^* \quad B_1^*) \tag{2.17}$$

$$U_v = -W_3^{-1} W_6, \tag{2.18}$$

and

$$U_n = (I_n \quad 0_n). \tag{2.19}$$

We define the matrices Q_1, Q_2, and Q_3 by

$$Q_1 = U_n^* W_1 U_n + U_n^* W_2 U_z + U_z^* W_2^* U_n + U_z^* W_3 U_z, \tag{2.20}$$

$$Q_2 = U_n^* W_2 U_v + U_z^* W_3 U_v + U_n^* W_4 + U_z^* W_6. \tag{2.21}$$

$$Q_3 = U_v^* W_3 U_v + W_5 + 2U_v^* W_6. \tag{2.22}$$

The system given by (2.12) can be written as

$$\dot{\zeta} = M(t)\zeta + N(t)v, \tag{2.23}$$

with

$$x(0) = 0, \;\; \theta(T) = 0, \tag{2.24}$$

and v needs to be selected to minimize the cost

$$\frac{\int_0^T \frac{1}{2} v^*(t) R(t) v(t)\, dt}{\int_0^T \{ \frac{1}{2} \zeta^*(t) Q_1(t) \zeta(t) + \zeta^* Q_2 v + \frac{1}{2} v^* Q_3 v \}\, dt}. \tag{2.25}$$

We now investigate the conditions under which a minimizing v exists. Note that if $v = 0$, the denominator of (2.25) is zero and that the minimum value of (2.25) over $v \neq 0$ is λ.

THEOREM 2.1. *Consider the system given by (2.23) and (2.24). Assume that there exists a $v \in L_2(0,T)$ for which $0 < \int_0^T \{ \frac{1}{2} \zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2} v^* Q_3 v \}\, dt < \infty$. Let*

$$\lambda = \inf_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v\, dt}{\int_0^T \{ \frac{1}{2} \zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2} v^* Q_3 v \}\, dt}. \tag{2.26}$$

Also assume that $R(t) - \lambda Q_3(t) \geq \alpha > 0$ *for all* $t \in [0,T]$ *and* $\begin{pmatrix} Q_1 & Q_2 \\ Q_2^* & 0 \end{pmatrix}$ *is positive semidefinite. Then there exists* $v_0 \in L_2(0,T)$ *which minimizes (2.25). Also, the minimum value of (2.25) is strictly positive.*

Proof. Since (2.25) is invariant under scaling of v, it suffices to consider only those v for which $\int_0^T \{\frac{1}{2}\zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2} v^* Q_3 v\}\, dt = 1$. Let $\{v_i\}$ be a sequence in $L_2(0,T)$ such that $\lim_{i\to\infty} \int_0^T \frac{1}{2} v_i^* R v_i\, dt = \lambda$ with $\int_0^T \{\frac{1}{2}\zeta_i^* Q_1 \zeta_i + \zeta^* Q_2 v_i + \frac{1}{2} v_i^* Q_3 v_i\}\, dt = 1$ for each i. Here ζ_i is the response of

$$\dot{\zeta}_i = M(t)\zeta_i + N(t) v_i \tag{2.27}$$

with

$$x_i(0) = 0, \quad \theta_i(T) = 0. \tag{2.28}$$

Since $\{v_i\}$ is bounded in $L_2(0,T)$, a subsequence, still denoted by $\{v_i\}$ converges weakly to some $v_0 \in L_2(0,T)$. Also we can select the subsequence such that $\{\zeta_i(0)\}$, $\{\int_0^T v_i^* Q_3 v_i\, dt\}$, $\{\int_0^T v_i^* v_i\, dt\}$ are convergent. Let $\zeta_i(0) \to \zeta^0$. We have

$$\zeta_i(t) = \zeta_i(0) + \int_0^t \Phi(t,\tau) N(\tau) v_i(\tau)\, d\tau, \tag{2.29}$$

where $\Phi(t,\tau)$ is the transition matrix of (2.27). Let $\zeta_0(t)$ satisfy

$$\dot{\zeta}_0 = M(t)\zeta_0 + N(t) v_0, \quad \zeta_0(0) = \zeta^0. \tag{2.30}$$

It is clear that by the weak convergence of $\{v_i\}$, $\zeta_i(t)$ converges pointwise to $\zeta_0(t)$. From (2.29), it follows that for sufficiently large i, the responses $\zeta_i(t)$ are uniformly bounded. By the Lebesgue dominated convergence theorem, we conclude that

$$\int_0^T \zeta_i^*(t) Q_1(t) \zeta_i(t)\, dt \to \int_0^T \zeta_0^*(t) Q_1(t) \zeta_0(t)\, dt. \tag{2.31}$$

Also we have

$$\int_0^T \zeta_i^* Q_2 v_i\, dt - \int_0^T \zeta_0^* Q_2 v_0\, dt = \int_0^T (\zeta_i - \zeta_0)^* Q_2 v_i\, dt + \int_0^T \zeta_0^* Q_2 (v_i - v_0)\, dt. \tag{2.32}$$

It can be easily shown that the right side of (2.32) goes to zero as $i \to \infty$. Thus

$$\lim_{i\to\infty} \int_0^T \zeta_i^* Q_2 v_i\, dt = \int_0^T \zeta_0^* Q_2 v_0\, dt. \tag{2.33}$$

Note that

$$\int_0^T \{\frac{1}{2}\zeta_i^* Q_1 \zeta_i + \zeta_i^* Q_2 v_i + \frac{1}{2} v_i^* Q_3 v_i\}\, dt = 1 \forall i, \tag{2.34}$$

$$\int_0^T \{\frac{1}{2}\zeta_0^* Q_1 \zeta_0 + \zeta_0^* Q_2 v_0 + \frac{1}{2} v_0^* Q_3 v_0\}\, dt = 1 - \lim_{i\to\infty} \int_0^T \frac{1}{2} v_i^* Q_3 v_i\, dt + \int_0^T \frac{1}{2} v_0^* Q_3 v_0\, dt. \tag{2.35}$$

We already noted that $\lim_{i\to\infty} \int_0^T \frac{1}{2} v_i^* Q_3 v_i\, dt$ exists. As a consequence of weak convergence

$$\int_0^T \frac{1}{2} v_0^* Q_3 v_0\, dt \le \lim_{i\to\infty} \int_0^T \frac{1}{2} v_i^* Q_3 v_i\, dt. \tag{2.36}$$

We claim that the right side of (2.35) is strictly positive. Otherwise by the positive semidefiniteness of $\begin{pmatrix} Q_1 & Q_2 \\ Q_2^* & 0 \end{pmatrix}$ and by (2.34), (2.35) and (2.36), it follows that $v_0 = 0$ and $\lim_{i\to\infty} \int_0^T \frac{1}{2} v_i^* Q_3 v_i\, dt = 1$. From the assumption that $R - \lambda Q_3 \ge \alpha > 0$, we have

$$\frac{1}{2} v_i^* R v_i \ge \lambda \frac{1}{2} v_i^* Q_3 v_i + \frac{\alpha}{2} v_i^* v_i. \tag{2.37}$$

Integrating both sides from 0 to T and letting i go to ∞, we get $\lambda \ge \lambda + \beta$, where $\beta > 0$. This contradiction shows that

$$\int_0^T \{\frac{1}{2}\zeta_0^* Q_1 \zeta_0 + \zeta_0^* Q_2 v_0 + \frac{1}{2} v_0^* Q_3 v_0\}\, dt > 0. \tag{2.38}$$

We now show that

$$\frac{\int_0^T \frac{1}{2} v_0^* R v_0\, dt}{\int_0^T \{\frac{1}{2}\zeta_0^* Q_1 \zeta_0 + \zeta_0^* Q_2 v_0 + \frac{1}{2} v_0^* Q_3 v_0\}\, dt} \le \lambda. \tag{2.39}$$

Since $\lim_{i\to\infty} \int_0^T \frac{1}{2} v_i^* R v_i\, dt = \lambda$, we only need to show that

$$\lim_{i\to\infty} \int_0^T \frac{1}{2} v_i^* R v_i\, dt - \frac{\int_0^T \frac{1}{2} v_0^* R v_0\, dt}{\int_0^T \{\frac{1}{2}\zeta_0^* Q_1 \zeta_0 + \zeta_0^* Q_2 v_i + \frac{1}{2} v_i^* Q_3 v_i\}\, dt} \ge 0. \tag{2.40}$$

Indeed the numerator of

$$\lim_{i\to\infty} \int_0^T \frac{1}{2} v_i^* Q_1 v_i\, dt - \frac{\int_0^T \frac{1}{2} v_0^* Q_1 v_0\, dt}{1 - \lim_{i\to\infty} \int_0^T \frac{1}{2} v_i^* Q_3 v_i\, dt + \int_0^T \frac{1}{2} v_0^* Q_3 v_0\, dt} \tag{2.41}$$

is given by

$$\lim_{i\to\infty}\int_0^T \frac{1}{2}v_i^*(R-\lambda Q_3)v_i\,dt - \int_0^T \frac{1}{2}v_0^*(R-\lambda Q_3)v_0\,dt. \tag{2.42}$$

Since $R-\lambda Q_3 > 0$, as a consequence of the weak convergence of $\{v_i\}$ to v_0, the above quantity is nonnegative.

Also λ needs to be greater than zero. Otherwise $v_0 = 0$, which implies that the corresponding unique optimal $u = 0$. This makes the response $\zeta_0(t) = 0$, and hence the denominator of (2.25) is zero, contradicting (2.38). □

3. INFIMAL H_∞ NORM

In the previous section, the minimax problem was converted into a minimization problem. The resulting system was

$$\dot{\zeta} = M(t)\zeta + N(t)v \tag{3.1}$$

with

$$x(0) = 0, \quad \theta(T) = 0, \tag{3.2}$$

where v needs to be selected to minimize the cost

$$\frac{\int_0^T \frac{1}{2}v^*(t)R(t)v(t)\,dt}{\int_0^T \{\frac{1}{2}\zeta^*(t)Q_1(t)\zeta(t) + \zeta^*(t)Q_2(t)v(t) + \frac{1}{2}v^*(t)Q_3(t)v(t)\}\,dt}. \tag{3.3}$$

We now state the conditions that are satisfied by an optimal $v(t)$.

THEOREM 3.1. *Consider the system given by (3.1)-(3.3). Assume that $R-\lambda Q_3$ is invertible for all $t\in[0,T]$. If $v_0(t)$ minimizes (3.3), then there exists a nonzero $\rho(t) = (\,p^*(t) \quad q^*(t)\,)^*$ such that*

$$\frac{d\rho}{dt} = -M^*\rho - \lambda Q_1\zeta - \lambda Q_2 v, \tag{3.4}$$

where $p(t)$ and $q(t)$ are components of the adjoint vector corresponding to $x(t)$ and $\theta(t)$ respectively, such that

$$\begin{aligned} x(0) = 0,\ \theta(T) = 0,\\ p(T) = 0,\ q(0) = 0, \end{aligned} \tag{3.5}$$

where

$$\lambda = \inf_{v\neq 0}\frac{\int_0^T \frac{1}{2}v^*Rv\,dt}{\int_0^T\{\frac{1}{2}\zeta^*Q_1\zeta + \zeta^*Q_2 v + \frac{1}{2}v^*Q_3 v\}\,dt}, \tag{3.6}$$

and

$$v_0(t) = (R-\lambda Q_3)^{-1}\{\lambda Q_2^*\zeta + N^*\rho\}. \tag{3.7}$$

Proof. If $v_0(t)$ minimizes (3.3), then it also minimizes

$$\tilde{J}(v) \triangleq \int_0^T \frac{1}{2} v^* R v \, dt - \lambda \int_0^T \{\frac{1}{2}\zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2} v^* Q_3 v\} \, dt. \tag{3.8}$$

The theorem now follows from the maximum principle[10]. □

Let

$$\tilde{M} = M + \lambda N (R - \lambda Q_3)^{-1} Q_2^*, \tag{3.9}$$

$$\tilde{N} = N (R - \lambda Q_3)^{-1} N^*, \tag{3.10}$$

and

$$\tilde{L} = -\lambda Q_1 - \lambda^2 Q_2 (R - \lambda Q_3)^{-1} Q_2^*. \tag{3.11}$$

The variables satisfy a two point boundary value problem given by

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\rho} \end{pmatrix} = \begin{pmatrix} \tilde{M} & \tilde{N} \\ \tilde{L} & -\tilde{M}^* \end{pmatrix} \begin{pmatrix} \zeta \\ \rho \end{pmatrix}. \tag{3.12}$$

with

$$\begin{aligned} x(0) = 0, \;\; \theta(T) = 0, \\ p(T) = 0, \;\; q(0) = 0. \end{aligned} \tag{3.13}$$

We now give a criterion for the estimation of λ. Notice that $\lambda = \min_{v \neq 0} \max_u J(u, v)$ and gives a measure of performance of the optimal controller under worst-case conditions corresponding to $v_0(t)$. In the H_∞ case, the evaluation of λ would entail the γ-iteration.

THEOREM 3.2. *Let λ be the smallest positive value for which the boundary value problem given by (3.12) and (3.13) has a solution (ζ, ρ) with $\int_0^T \{\frac{1}{2}\zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2} v^* Q_3 v\} \, dt > 0$, where $v \triangleq (R - \lambda Q_3)^{-1}\{\lambda Q_2^* \zeta + N^* \rho\}$. Then λ is the minimum value of (3.12), (ζ, ρ) is an optimal pair and $v = (R - \lambda Q_3)^{-1}\{\lambda Q_2^* \zeta + N^* \rho\}$ is the worst exogenous input.*

Proof. It is clear from Theorem 3.1 that if $v_0(t)$ minimizes (3.3), then it satisfies (3.12) and (3.13), with λ being the minimum value of (3.3). Now suppose (ζ, ρ) satisfies (3.12) and (3.13) for some λ. Let $v = (R - \lambda Q_3)^{-1}\{\lambda Q_2^* \zeta + N^* \rho\}$. In the following equations (,) denotes an inner product.

We have

$$\int_0^T ((R - \lambda Q_3) v, v) \, dt = \int_0^T (\lambda Q_2^* \zeta, v) \, dt + \int_0^T (N^* \rho, v) \, dt. \tag{3.14}$$

By equation (3.1), the second integral of (3.14) can be written as

$$\int_0^T (N^* \rho, v) \, dt = \int_0^T (\rho, N v) \, dt = \int_0^T (\rho, \dot{\zeta} - M\zeta) \, dt. \tag{3.15}$$

An integration by parts and equations (3.4) and (3.5) yield

$$\int_0^T (\rho, \dot{\zeta} - M\zeta)\, dt = \lambda \int_0^T (Q_1\zeta, \zeta)\, dt + \lambda \int_0^T (\zeta, Q_2 v)\, dt. \tag{3.16}$$

Substituting (3.16) in (3.14), we get

$$\int_0^T v^* R v\, dt = \lambda \int_0^T \{\zeta^* Q_1 \zeta + 2\zeta^* Q_2 v + v^* Q_3 v\}\, dt. \tag{3.17}$$

Thus, the cost associated with v is λ. Hence, if (ζ, ρ) is a nontrivial solution of the boundary value problem given by (3.12) and (3.13) for the smallest parameter $\lambda > 0$ with $\int_0^T \{\frac{1}{2}\zeta^* Q_1 \zeta + \zeta^* Q_2 v + \frac{1}{2} v^* Q_3 v\}\, dt > 0$, then λ is the optimal value and (ζ, ρ) is an optimal pair. □

Note that the boundary value problem (3.12)-(3.13) has a solution with a nonvanishing denominator for (2.4) for at most a countably infinite values of λ. Theorem 3.2 gives a sufficient condition for an exogenous input to be optimal. Equations (3.12)-(3.13) and Theorem 3.2 completely characterize the worst exogenous input.

The criterion in Theorem 3.2 can be used to devise computational tools for the evaluation of the infimal H_∞ norm in the finite horizon case. Our computational experience shows that for time-invariant problems, the infimal H_∞ norm in the finite horizon case approaches that in the infinite horizon case as the final time T becomes large.

4. COMPUTATION OF λ

Making use of the transition matrix, the solution of (3.12) can be expressed as

$$\begin{pmatrix} x(t) \\ \theta(t) \\ p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} \phi_{11}(t,0) & \phi_{12}(t,0) & \phi_{13}(t,0) & \phi_{14}(t,0) \\ \phi_{21}(t,0) & \phi_{22}(t,0) & \phi_{23}(t,0) & \phi_{24}(t,0) \\ \phi_{31}(t,0) & \phi_{32}(t,0) & \phi_{33}(t,0) & \phi_{34}(t,0) \\ \phi_{41}(t,0) & \phi_{42}(t,0) & \phi_{43}(t,0) & \phi_{44}(t,0) \end{pmatrix} \begin{pmatrix} x(0) \\ \theta(0) \\ p(0) \\ q(0) \end{pmatrix}. \tag{4.1}$$

The boundary conditions given by (3.13) yield

$$\begin{pmatrix} \phi_{22}(T,0) & \phi_{23}(T,0) \\ \phi_{32}(T,0) & \phi_{33}(T,0) \end{pmatrix} \begin{pmatrix} x(0) \\ \theta(0) \end{pmatrix} = 0. \tag{4.2}$$

Let

$$\tilde{\phi} = \begin{pmatrix} \phi_{22} & \phi_{23} \\ \phi_{32} & \phi_{33} \end{pmatrix}. \tag{4.3}$$

In view of (4.2) and (3.12)-(3.13), we have $\det(\tilde{\phi}(T,0)) = 0$ if and only if the solution (ζ, ρ) of (3.12)-(3.13) is not identically zero. Thus, we need the least positive λ which makes

$\det\big(\tilde{\phi}(T,0)\big) = 0$ and the denominator of (2.4) positive. This can be obtained by doing a search with λ over an interval on which there is a change in the sign of the determinant.

We found the following algorithm to be numerically more stable since numbers of lesser magnitude are involved in the computation of the transition matrices in (4.4). We have

$$\begin{pmatrix} \zeta(T) \\ \rho(T) \end{pmatrix} = \phi(T,\frac{T}{2})\phi(\frac{T}{2},0)\begin{pmatrix} \zeta(0) \\ \rho(0) \end{pmatrix}. \tag{4.4}$$

Let

$$\phi^{-1}(T,\frac{T}{2}) = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} \end{pmatrix}, \tag{4.5}$$

and

$$\phi(\frac{T}{2},0) = \begin{pmatrix} \nu_{11} & \nu_{12} & \nu_{13} & \nu_{14} \\ \nu_{21} & \nu_{22} & \nu_{23} & \nu_{24} \\ \nu_{31} & \nu_{32} & \nu_{33} & \nu_{34} \\ \nu_{41} & \nu_{42} & \nu_{43} & \nu_{44} \end{pmatrix}. \tag{4.6}$$

Making use of $x(0) = q(0) = \theta(T) = p(T) = 0$, we have

$$\begin{pmatrix} \xi_{11} & \xi_{14} \\ \xi_{21} & \xi_{24} \\ \xi_{31} & \xi_{34} \\ \xi_{41} & \xi_{44} \end{pmatrix}\begin{pmatrix} x(T) \\ q(T) \end{pmatrix} = \begin{pmatrix} \nu_{12} & \nu_{13} \\ \nu_{22} & \nu_{23} \\ \nu_{32} & \nu_{33} \\ \nu_{42} & \nu_{43} \end{pmatrix}\begin{pmatrix} \theta(0) \\ p(0) \end{pmatrix}. \tag{4.7}$$

The above equation has a nontrivial solution if and only if

$$\det\begin{pmatrix} \xi_{11} & \xi_{14} & \nu_{12} & \nu_{13} \\ \xi_{21} & \xi_{24} & \nu_{22} & \nu_{23} \\ \xi_{31} & \xi_{34} & \nu_{32} & \nu_{33} \\ \xi_{41} & \xi_{44} & \nu_{42} & \nu_{43} \end{pmatrix} = 0. \tag{4.8}$$

Thus, we need the least positive λ which makes the above determinant zero.

5. A Differential Equation for λ

For simplicity we derive a differential equation for λ only in the case where $W_4 = W_5 = W_6 = 0$. Note that this makes $Q_2 = Q_3 = 0$. Thus equations (3.12) and (3.13) can be written as

$$\begin{aligned} \dot{\zeta} &= M\zeta + NR^{-1}N^*\rho \\ \dot{\rho} &= -\lambda Q_1\zeta - M^*\rho \end{aligned} \tag{5.1}$$

with

$$\begin{aligned} x(0) = 0, \theta(T) = 0, \\ p(T) = 0, q(0) = 0. \end{aligned} \tag{5.2}$$

Now assume that the final time is changed to $T + \Delta T$ where ΔT is an elemental increment. The solution of the above boundary value problem can be extended to $[0, T + \Delta T]$. Suppose ζ_1 and ρ_1 are the elemental variations in (ζ, ρ) owing to the increment ΔT in T. That is, $(\zeta + \zeta_1, \rho + \rho_1)$ is the new optimal pair. Also denote the variation in λ by $\Delta\lambda$. We have

$$\begin{aligned} \dot{\zeta}_1 &= M\zeta_1 + NR^{-1}N^*\rho_1 \\ \dot{\rho}_1 &= -\lambda Q_1\zeta_1 - M^*\rho_1 - \Delta\lambda Q_1\zeta \end{aligned} \tag{5.3}$$

with

$$\begin{aligned} x_1(0) = 0, \theta_1(T + \Delta T) = -\theta(T + \Delta T), \\ p_1(T + \Delta T) = -p(T + \Delta T), q_1(0) = 0. \end{aligned} \tag{5.4}$$

THEOREM 5.1. *As a function of* T, λ *satisfies*

$$\frac{d\lambda}{dT} = \frac{-\lambda\zeta^*(T)Q_1(T)\zeta(T) - 2\zeta^*(T)M^*(T)\rho(T) - \rho^*(T)N(T)R^{-1}(T)N^*(T)\rho(T)}{\int_0^T \zeta^* Q_1 \zeta \, dt}. \tag{5.5}$$

Proof: From (5.3),

$$\int_0^{T+\Delta T} \zeta^* \dot{\rho}_1 \, dt = -\int_0^{T+\Delta T} \{\lambda\zeta^*Q_1\zeta_1 + \zeta^*M^*\rho_1 + \Delta\lambda\zeta^*Q_1\zeta\} \, dt. \tag{5.6}$$

By an integration by parts,

$$\int_0^{T+\Delta T} \zeta^* \dot{\rho}_1 \, dt = \zeta^*\rho_1(T + \Delta T) - \int_0^{T+\Delta T} \{\zeta^*M^*\rho_1 + \rho^*NR^{-1}N^*\rho_1\} \, dt. \tag{5.7}$$

From (5.6) and (5.7),

$$\int_0^{T+\Delta T} \{\lambda\zeta^*Q_1\zeta_1 + \Delta\lambda\zeta^*Q_1\zeta\} \, dt = -\zeta^*\rho_1(T + \Delta T) + \int_0^{T+\Delta T} \rho^*NR^{-1}N^*\rho_1 \, dt. \tag{5.8}$$

From (5.1), the first integral on the left side of (5.8) can be written as

$$\int_0^{T+\Delta T} \lambda\zeta^*Q_1\zeta_1 \, dt = -\int_0^{T+\Delta T} (\dot{\rho} + M^*\rho)^*\zeta_1 \, dt. \tag{5.9}$$

Integrating the right side of (5.9) by parts, we get

$$\int_0^{T+\Delta T} \lambda\zeta^* Q_1 \zeta_1 \, dt = -\rho^* \zeta_1(T+\Delta T) + \int_0^{T+\Delta T} \rho^* N R^{-1} N^* \rho \, dt. \tag{5.10}$$

Substituting (5.10) in (5.8), we get

$$\Delta\lambda \int_0^{T+\Delta T} \zeta^* Q_1 \zeta \, dt = \rho^*(T+\Delta T)\zeta_1(T+\Delta T) - \zeta^*(T+\Delta T)\rho_1(T+\Delta T). \tag{5.11}$$

We have

$$\begin{aligned} \rho^*\zeta_1(T+\Delta T) - \zeta^*\rho_1(T+\Delta T) &= \rho^*(T)\zeta_1(T+\Delta T) - \zeta^*(T)\rho_1(T+\Delta T) + o(\Delta T) \\ &= -q^*(T)\theta(T+\Delta T) + x^*(T)p(T+\Delta T) + o(\Delta T) \\ &= -q^*(T)\dot{\theta}(T)\Delta T + x^*(T)\dot{p}(T)\Delta T + o(\Delta T) \\ &= -\rho^*(T)\dot{\zeta}(T)\Delta T + \zeta^*(T)\dot{\rho}(T)\Delta T + o(\Delta T) \end{aligned} \tag{5.12}$$

From (5.11) and (5.12), we get (5.5). □

6. EXAMPLES

The above theory is useful in the computation of the infimal H_∞ norm. This problem is still being researched actively in the case of both static and dynamic controllers and there are a variety of algorithms in the literature. In the examples given below, we compute the minimum H_∞ norm given by (2.4) as the final time T varies. The programs were written using PC-MATLAB and the least positive λ which satisfies equation (4.8) was found. The infimal finite-time H_∞ norm γ is $1/\sqrt{\lambda}$.

EXAMPLE 1. We consider the tracking example from [11]. In this case

$$A = \begin{pmatrix} -0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$W_1 = \begin{pmatrix} 0.0081 & -0.045 & -0.045 \\ -0.045 & 0.25 & 0.25 \\ -0.045 & 0.25 & 0.25 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad W_3 = 1$$

$$W_4 = \begin{pmatrix} 0.009 \\ -0.05 \\ -0.05 \end{pmatrix}, \quad W_5 = 0.01, \quad W_6 = 0, \quad R = 1.$$

The results are given below.

TABLE 1: Results of Example 1		
T	λ	$\gamma = 1/\sqrt{\lambda}$
5	17.6023	0.2384
10	16.3113	0.2476
15	15.9802	0.2502
20	15.7944	0.2516
25	15.7442	0.2520
30	15.7224	0.2522

EXAMPLE 2. This example, taken from [12] has

$$A = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 \\ -1 & 0 & -2 & -3 \\ -2 & -1 & 2 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

with

$$R = 1, \quad W_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and zero entries in W_2, W_4, W_5, and W_6. Table 2 gives the numerical results.

TABLE 2: Results of Example 2		
T	λ	$\gamma = 1/\sqrt{\lambda}$
5	3.8975	0.5065
10	0.9220	1.0414
15	0.7001	1.1951
20	0.6722	1.2197
25	0.6681	1.2234

EXAMPLE 3. The last example is also taken from [12]. In this case

$$A = \begin{pmatrix} 0 & 1 & 4 & -4 & 1 \\ -3 & -1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 2 & 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & -2 & -2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 4 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

with

$$R = 1, \quad W_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with the rest of the matrices having zero entries. The results are given in Table 3.

TABLE 3: Results of Example 3

T	λ	$\gamma = 1/\sqrt{\lambda}$
5	0.89262204	1.05843988
10	0.87538735	1.06880842
15	0.87477568	1.06918203
18	0.87476217	1.06919029

7. Suboptimal Problem Formulation

The main contribution of the remaining portion of the report is the derivation of the suboptimal controller in the time-varying case for a general performance index. We consider a generalized finite horizon suboptimal H_∞ problem. An expression for a state feedback controller is given in terms of solution of a dynamic Riccati equation. Also an expression for a suboptimal output feedback controller is developed in terms of solutions of two dynamic Riccati equations. Throughout the report an objective has been to derive results in as general a case as feasible. The formulae for the synthesis of the suboptimal H_∞ controller are summarized in Section 5 and these can be programmed easily on a digital computer to synthesize a suboptimal H_∞ controller. In the time invariant case, if the final time is sufficiently large, the solutions of the dynamic Riccati equations converge to the solutions of the corresponding algebraic Riccati equations.

Let the n-dimensional time-varying system be given by

$$\dot{x} = A(t)x + B_1(t)u + B_2(t)v, \quad x(t_0) = 0, \tag{7.1}$$

$$z = C(t)x + D(t)u + E(t)v, \tag{7.2}$$

$$y = C_2(t)x + D_2(t)u + E_2(t)v. \tag{7.3}$$

Without loss of generality, let $t_0 = 0$. Also let

$$\lambda_{opt} = \max_{u} \min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v \, dt}{\int_0^T \frac{1}{2} z^* W z \, dt}, \tag{7.4}$$

where R and W are assumed to be positive definite and the superscript * denotes a matrix or vector transpose. Computational techniques for the evaluation of λ_{opt} are given in Section 4. The problems addressed in the report can be stated as follows.

Problem 1. Given $\lambda < \lambda_{opt}$, find a full state feedback controller, if it exists, for which

$$\min_{v\neq 0} \frac{\int_0^T \frac{1}{2} v^* R v \, dt}{\int_0^T \frac{1}{2} z^* W z \, dt} > \lambda.$$

Problem 2. Given $\lambda < \lambda_{opt}$, find an output feedback controller, if it exists, for which

$$\min_{v\neq 0} \frac{\int_0^T \frac{1}{2} v^* R v \, dt}{\int_0^T \frac{1}{2} z^* W z \, dt} > \lambda.$$

8. Full State Feedback Problem

Consider the performance criterion

$$\int_0^T \frac{1}{2} v^* R v \, dt - \lambda \int_0^T \frac{1}{2} z^* W z \, dt. \tag{8.1}$$

We will first find a saddle point (u^0, v^0) with respect to the criterion (8.1). The motivation for finding the saddle point is so that we can construct a suboptimal H_∞ controller on the finite interval $[0, T]$.

The functional (8.1) can be written as

$$J(u,v) = \int_0^T \frac{1}{2} v^* R v \, dt - \lambda \int_0^T \{\frac{1}{2} x^* W_1 x + x^* W_2 u + \frac{1}{2} u^* W_3 u$$
$$+ x^* W_4 v + \frac{1}{2} v^* W_5 v + u^* W_6 v\} \, dt. \tag{8.2}$$

Given $u(t)$, let v^0 maximize (8.2). The following lemma characterizes $v^0(t)$.

LEMMA 8.1. *Let λ be such that $R - \lambda W_5$ is positive definite for all $t \in [0, T]$. For a given u, if $v^0(t)$ minimizes (8.2), then there exists a nonzero $\eta(t)$ such that*

$$\frac{d\eta}{dt} = -A^* \eta - \lambda W_1 x - \lambda W_2 u - \lambda W_4 v^0, \quad \eta(T) = 0, \tag{8.3}$$

and

$$v^0(t) = (R - \lambda W_5)^{-1} \{B_2^* \eta + \lambda W_4^* x + \lambda W_6^* u.\} \tag{8.4}$$

Proof. By the maximum principle[10], there exists an adjoint response $\eta(t)$ such that the Hamiltonian

$$H = \lambda\{\frac{1}{2} x^* W_1 x + x^* W_2 u + \frac{1}{2} u^* W_3 u + x^* W_4 v + \frac{1}{2} v^* W_5 v + u^* W_6 v\}$$
$$- \frac{1}{2} v^* R v + \eta^* \{A(t) x + B_1(t) u + B_2(t) v.\}. \tag{8.5}$$

is maximized almost everywhere on $[0, T]$. Satisfaction of $\frac{\partial H}{\partial v} = 0$ yields (8.4). The adjoint variable η satisfies

$$\frac{d\eta}{dt} = -\frac{\partial H}{\partial x} = -A^*\eta - \lambda W_1 x - \lambda W_2 u - \lambda W_4 v^0. \tag{8.6}$$

By the transversality condition, $\eta(T) = 0$. □

In a similar manner, we can get an expression for an optimal $u^0(t)$ which maximizes (8.2) for given v and λ.

LEMMA 8.2. *Consider the system given by (7.1). Assume that W_3 is positive definite for all $t \in [0, T]$. For a given v, if $u^0(t)$ maximizes (8.2), then there exists a nonzero $\psi(t)$ such that*

$$\frac{d\psi}{dt} = -A^*\psi + W_1 x + W_2 u^0 + W_4 v, \quad \psi(T) = 0, \tag{8.7}$$

and

$$u^0(t) = W_3^{-1}\{B_1^*\psi - W_2^* x - W_6 v\}. \tag{8.8}$$

Proof. Proof is similar to that of Lemma 8.1. □

Simultaneous solution of (8.3), (8.4), (8.7), and (8.8) yields a saddle point solution (u^0, v^0) for the functional given by (8.2). We now express the above minimax solution in a simpler form.

From (8.3) and (8.7), at a saddle point solution (u^0, v^0), we get

$$\frac{d}{dt}(\lambda\psi + \eta) = -A^*(t)(\lambda\psi + \eta), \quad \lambda\psi(T) + \eta(T) = 0. \tag{8.9}$$

It follows that

$$\lambda\psi(t) + \eta(t) = 0, \quad t \in [0, T]. \tag{8.10}$$

Thus the saddle point solution is characterized by

$$\frac{d\psi}{dt} = -A^*\psi + W_1 x + W_2 u^0 + W_4 v^0, \tag{8.11}$$

$$u^0(t) = W_3^{-1}\{B_1^*\psi - W_2^* x - W_6^* v^0\}, \tag{8.12}$$

$$v^0(t) = \lambda(R - \lambda W_5)^{-1}\{-B_2^*\psi + W_4^* x + W_6^* u^0\}. \tag{8.13}$$

We define the *full state feedback controller* the following way. Assuming that the inverse of $W_3 + \lambda W_6(R - \lambda W_5)^{-1} W_6^*$ exists, let

$$\Omega = (R - \lambda W_5)^{-1}, \tag{8.14}$$

$$\Lambda = \{W_3 + \lambda W_6 \Omega W_6^*\}^{-1}, \tag{8.15}$$

$$U_1 = \Lambda(B_1^* + \lambda W_6 \Omega B_2^*), \tag{8.16}$$

$$U_2 = -\Lambda(W_2^* + \lambda W_6 \Omega W_4^*), \tag{8.17}$$

$$V_1 = \lambda\Omega(-B_2^* + W_6^* U_1), \tag{8.18}$$

$$V_2 = \lambda\Omega(W_4^* + W_6^* U_2). \tag{8.19}$$

Substituting (8.13) in (8.12), we get

$$u^0 = U_1\psi + U_2 x. \tag{8.20}$$

Substituting (8.20) in (8.13), we get

$$v^0 = V_1\psi + V_2 x. \tag{8.21}$$

Let ϵ be arbitrary. On $[\epsilon, T]$, let

$$\psi(t) = P(t)x(t). \tag{8.22}$$

On $[\epsilon, T]$, we get

$$\begin{aligned}\dot{P} &+ P(A + B_1U_2 + B_2V_2) + (A^* - W_2U_1 - W_4V_1)P \\ &+ P(B_1U_1 + B_2V_1)P - (W_1 + W_2U_2 + W_4V_2) = 0, \quad P(T) = 0.\end{aligned} \tag{8.23}$$

Define the feedback controller and the exogenous input by

$$u_0 = U_1Px + U_2x, \tag{8.24}$$

$$v_0 = V_1Px + V_2x. \tag{8.25}$$

We now show that the performance of this feedback controller is greater than λ.

THEOREM 8.1. *Consider equations (8.23) and (8.24). Then for this controller*

$$\inf_{v\neq 0} \frac{\int_0^T \frac{1}{2}v^*Rv\,dt}{\int_0^T \frac{1}{2}z^*Wz\,dt} > \lambda. \tag{8.26}$$

Proof. An elementary calculation shows that in equation (8.23), $B_1U_1 + B_2V_1$ and $W_1 + W_2U_2 + W_4V_2$ are symmetric and

$$(A + B_1U_2 + B_2V_2)^* = A^* - W_2U_1 - W_4V_1. \tag{8.27}$$

Thus P is symmetric. Since (8.20) and (8.21) follow from (8.12) and (8.13), we have

$$u_0 = W_3^{-1}\{B_1^*Px - W_2^*x - W_6v_0\}, \tag{8.28}$$

$$v_0 = \lambda\Omega\{-B_2^*Px + W_4^*x + W_6^*u_0\}. \tag{8.29}$$

Since $P(T) = x(0) = 0$, we have

$$\int_0^T \frac{d}{dt}(x^*Px)\,dt = 0 \tag{8.30}$$

Since

$$\dot{x} = Ax + B_1u_0 + B_2v = (A + B_1U_1P + B_1U_2)x + B_2v, \tag{8.31}$$

where v is an arbitrary function of time, we get after some algebra

$$\begin{aligned}\frac{d}{dt}(x^*Px) &= x^*\dot{P}x + 2x^*P\dot{x} \\ &= x^*W_1x + x^*W_2u_0 + x^*W_4v_0 - x^*PB_2v_0 + x^*PB_1u_0 + 2x^*PB_2v.\end{aligned} \tag{8.32}$$

From (8.28) and (8.29), we get

$$B_1^*Px = W_3u_0 + W_2^*x + W_6v_0, \tag{8.33}$$

$$B_2^*Px = -\frac{\Omega^{-1}v_0}{\lambda} + W_4^*x + W_6u_0, \tag{8.34}$$

Substituting (8.33) and (8.34) in (8.32), we get

$$\begin{aligned}\frac{d}{dt}(x^*Px) &= x^*W_1x + 2x^*W_2u_0 + u_0^*W_3u_0 + 2x^*W_4v \\ &\quad -\frac{v^*\Omega^{-1}v}{\lambda} + 2u_0^*W_6v + \frac{(v-v_0)^*\Omega^{-1}(v-v_0)}{\lambda}.\end{aligned} \tag{8.35}$$

From (8.35) and (8.30), we get

$$\int_0^T v^*Rv\,dt - \lambda\int_0^T z^*Wz\,dt = \int_0^T (v-v_0)^*\Omega^{-1}(v-v_0)\,dt. \tag{8.36}$$

Note that in the above equation v is an open loop function of t, whereas v_0 is a feedback function of x. Note also that Ω^{-1} is positive definite on $[0,T]$. Since the map $v \to v - v_0$ and its inverse are bounded, it follows that there exists $\delta > 0$ such that

$$\int_0^T (v-v_0)^*\Omega^{-1}(v-v_0)\,dt \geq \delta\int_0^T v^*Rv\,dt. \tag{8.37}$$

From (8.36) and (8.37), we have (8.26). □

9. Output Feedback Controller

Consider again the system given by

$$\dot{x} = A(t)x + B_1(t)u + B_2(t)v, \tag{9.1}$$

$$z = C(t)x + D(t)u + E(t)v, \tag{9.2}$$

$$y = C_2(t)x + D_2(t)u + E_2(t)v. \tag{9.3}$$

Let

$$\tilde{C} = C_2 + E_2 V_1 P + E_2 V_2, \tag{9.4}$$

where V_1 and V_2 are defined by (8.18) and (8.19). Assume that the controller is of the form

$$\dot{q} = Aq + B_1(U_1 Pq + U_2 q) + B_2(V_1 Pq + V_2 q) + L(\tilde{C}q + D_2 u - y), \tag{9.5}$$

$$u = U_1 Pq + U_2 q, \tag{9.6}$$

where the observer gain L needs to be determined.

Let

$$u_0 = U_1 Px + U_2 x, \tag{9.7}$$

$$v_0 = V_1 Px + V_2 x, \tag{9.8}$$

$$r = u - u_0, \tag{9.9}$$

$$w = v - v_0, \tag{9.10}$$

$$e = x - q. \tag{9.11}$$

Let the control be given by (9.6) and let P be the solution of (8.23). We have

$$\dot{x} = Ax + B_1(U_1 P + U_2)q + B_2 v, \tag{9.12}$$

where v is an arbitrary function of time.

LEMMA 9.1. *For the above system, we have*

$$\int_0^T v^* R v \, dt - \lambda \int_0^T z^* W z \, dt = \int_0^T w^* \Omega^{-1} w \, dt - \lambda \int_0^T r^* W_3 r \, dt - 2\lambda \int_0^T r^* W_6 w \, dt. \tag{9.13}$$

Proof. We have

$$x^* \dot{P} x + 2x^* P \dot{x} = x^* W_1 x + x^* W_2 u_0 + x^* W_4 v_0 + x^* PB_1(2u - u_0) + x^* PB_2(2v - v_0). \tag{9.14}$$

From (8.33) and (8.34),

$$B_1^* Px = W_3 u_0 + W_2^* x + W_6 v_0, \tag{9.15}$$

$$B_2^* Px = -\frac{\Omega^{-1} v_0}{\lambda} + W_4^* x + W_6^* u_0. \tag{9.16}$$

Incorporating (9.15) and (9.16) in (9.14) and rearranging, the right side of (9.14) can be written as

$$x^*W_1x + 2x^*W_2u + u^*W_3u + 2x^*W_4v - \frac{v^*\Omega^{-1}v}{\lambda} + 2u^*W_6v$$
$$-(u-u_0)^*W_3(u-u_0) + \frac{(v-v_0)^*\Omega^{-1}(v-v_0)}{\lambda} - 2(u-u_0)^*W_6(v-v_0). \qquad (9.17)$$

Since

$$\int_0^T \frac{d}{dt}(x^*Px)\,dt = \int_0^T \{x^*\dot{P}x + 2x^*P\dot{x}\}\,dt = 0, \qquad (9.18)$$

the result follows. □

We want to ultimately show that the right side of (9.13) is larger than $\tilde{\delta}\int_0^T v^*Rv\,dt$ for some $\tilde{\delta} > 0$. It easily follows that $\epsilon = x - q$ and $r = u - u_0$ satisfy

$$\dot{\epsilon} = (\tilde{A} + L\tilde{C})\epsilon + (B_2 + LE_2)w, \qquad (9.19)$$
$$r = \tilde{B}\epsilon, \qquad (9.20)$$

where

$$\tilde{A} = A + B_2V_1P + B_2V_2, \qquad (9.21)$$
$$\tilde{B} = -(U_1P + U_2). \qquad (9.22).$$

Note that $\tilde{C}$ is defined by (9.4) and L is the gain of the observer. The right side of (9.13) can be written as

$$\int_0^T w^*Rw\,dt - \lambda\int_0^T z_1^*Wz_1\,dt, \qquad (9.23)$$

where (see (7.2))

$$z_1 = Dr + Ew = D\tilde{B}\epsilon + Ew. \qquad (9.24)$$

We determine L by considering the dual of (9.19) and (9.24). Our ultimate goal is to show that for the L to be chosen, the controller given by (9.5) and (9.6) is suboptimal.

Let $\tau = -t$. The dual system can be defined on $[-T, 0]$ as

$$\frac{d\tilde{\epsilon}}{d\tau} = \tilde{A}^*\tilde{\epsilon} + \tilde{C}^*\tilde{u} + \tilde{B}^*D^*\tilde{w}, \qquad (9.25a)$$
$$\tilde{z}_1 = B_2^*\tilde{\epsilon} + E_2^*\tilde{u} + E^*\tilde{w}, \qquad (9.25b)$$

and the functional corresponding to (9.23) is written as

$$\int_{-T}^0 \frac{1}{2}\tilde{w}^*W^{-1}\tilde{w}\,d\tau - \lambda\int_{-T}^0 \frac{1}{2}\tilde{z}_1^*R^{-1}\tilde{z}_1\,d\tau. \qquad (9.26)$$

Let us write (9.26) as

$$\int_{-T}^{0} \frac{1}{2}\tilde{w}^* W^{-1}\tilde{w}\, d\tau - \lambda \int_{-T}^{0} \{\frac{1}{2}\tilde{\epsilon}^* \tilde{W}_1 \tilde{\epsilon} + \tilde{\epsilon}^* \tilde{W}_2 \tilde{u} + \frac{1}{2}\tilde{u}^* \tilde{W}_3 \tilde{u}$$
$$+\tilde{\epsilon}^* \tilde{W}_4 \tilde{w} + \frac{1}{2}\tilde{w}^* \tilde{W}_5 \tilde{w} + \tilde{u}^* \tilde{W}_6 \tilde{w}\}\, d\tau. \tag{9.27}$$

We now find a saddle point solution for the functional in (9.27). The saddle point solution will be useful in defining the gain L of the observer.

This problem is completely analogous to the full state feedback problem of Section 3 and we can write the solution by inspection. Equations (9.25a) and (9.27) are analogous to equations (7.1) and (8.2) respectively. Assume that $\tilde{W}_3$ and $W^{-1} - \lambda\tilde{W}_5$ are positive definite and β is the adjoint variable. Observing (8.11)-(8.13), the saddle point solution is characterized by

$$\frac{d\beta}{d\tau} = -\tilde{A}\beta + \tilde{W}_1\tilde{\epsilon} + \tilde{W}_2\tilde{u} + \tilde{W}_4\tilde{w}, \quad \beta(0) = 0, \tag{9.28}$$
$$\tilde{u} = \tilde{W}_3^{-1}\{\tilde{C}\beta - \tilde{W}_2^*\tilde{\epsilon} - \tilde{W}_6^*\tilde{w}\}. \tag{9.29}$$
$$\tilde{w} = \lambda(W^{-1} - \lambda\tilde{W}_5)^{-1}\{-D\tilde{B}\beta + \tilde{W}_4^*\tilde{\epsilon} + \tilde{W}_6^*\tilde{u}\}. \tag{9.30}$$

Assume that $\tilde{W}_3 + \lambda\tilde{W}_6(W^{-1} - \lambda\tilde{W}_5)^{-1}\tilde{W}_6^*$ is invertible. The equations analogous to (8.14)-(8.19) are

$$\Phi = (W^{-1} - \lambda\tilde{W}_5)^{-1}, \tag{9.31}$$
$$\Gamma = \{\tilde{W}_3 + \lambda\tilde{W}_6\Phi\tilde{W}_6^*\}^{-1}, \tag{9.32}$$
$$S_1 = \Gamma(\tilde{C} + \lambda\tilde{W}_6\Phi D\tilde{B}), \tag{9.33}$$
$$S_2 = -\Gamma(\tilde{W}_2^* + \lambda\tilde{W}_6\Phi\tilde{W}_4^*), \tag{9.34}$$
$$T_1 = \lambda\Phi(-D\tilde{B} + \tilde{W}_6^* S_1), \tag{9.35}$$
$$T_2 = \lambda\Phi(\tilde{W}_4^* + \tilde{W}_6^* S_2). \tag{9.36}$$

Substituting (9.30) in (9.29), we can write $\tilde{u}$ and $\tilde{w}$ as

$$\tilde{u} = S_1\beta + S_2\tilde{\epsilon}, \tag{9.37}$$
$$\tilde{w} = T_1\beta + T_2\tilde{\epsilon}. \tag{9.38}$$

Letting

$$\beta = Y\tilde{\epsilon} \tag{9.39}$$

on $[-T, 0]$, we get the Riccati equation

$$-\frac{dY}{d\tau} = (\tilde{A} - \tilde{W}_2 S_1 - \tilde{W}_4 T_1)Y + Y(\tilde{A}^* + \tilde{C}^* S_2 + \tilde{B}^* D^* T_2)$$
$$+Y(\tilde{C}^* S_1 + \tilde{B}^* D^* T_1)Y - (\tilde{W}_1 + \tilde{W}_2 S_2 + \tilde{W}_4 T_2), \quad Y(0) = 0. \tag{9.40}$$

which is analogous to (8.23). It can be easily verified that (9.40) is symmetric.

Define $\tilde{u}_0$ and $\tilde{w}_0$ by

$$\tilde{u}_0 = S_1 Y \tilde{\epsilon} + S_2 \tilde{\epsilon}, \tag{9.41}$$

$$\tilde{w}_0 = T_1 Y \tilde{\epsilon} + T_2 \tilde{\epsilon}. \tag{9.42}$$

LEMMA 9.2. *Consider equations (9.40) and (9.41) and assume that at $\tilde{u} = \tilde{u}_0$, given $\epsilon > 0$ there exists $\tilde{T} > 0$ such that for any $\tilde{w} \neq 0$ and all $T \geq \tilde{T}$, $|\tilde{\epsilon}^*(-T)Y(-T)\tilde{\epsilon}(-T)| \leq \epsilon \int_{-T}^{0} \tilde{w}^* W^{-1} \tilde{w}\, d\tau$. Then there exists $\delta > 0$ such that for all $T \geq \tilde{T}$,*

$$\int_{-T}^{0} \tilde{w}^* W^{-1} \tilde{w}\, d\tau - \lambda \int_{-T}^{0} \tilde{z}_1^* R^{-1} \tilde{z}_1\, d\tau \geq \delta \int_{-T}^{0} \tilde{w}^* W^{-1} \tilde{w}\, d\tau. \tag{9.43}$$

Proof. Following similar reasoning as in the proof of Theorem 8.1, we can write the equation analogous to (8.35) as

$$\begin{aligned} \frac{d}{d\tau}(\tilde{\epsilon}^* Y \tilde{\epsilon}) = {} & \tilde{\epsilon}^* \tilde{W}_1 \tilde{\epsilon} + 2\tilde{\epsilon}^* \tilde{W}_2 \tilde{u}_0 + \tilde{u}_0^* \tilde{W}_3 \tilde{u}_0 + 2\tilde{\epsilon}^* \tilde{W}_4 \tilde{w} + \tilde{w}^* \tilde{W}_5 \tilde{w} \\ & + 2\tilde{u}_0^* \tilde{W}_6 \tilde{w} - \frac{\tilde{w}^* W^{-1} \tilde{w}}{\lambda} + \frac{(\tilde{w} - \tilde{w}_0)^* \Phi^{-1} (\tilde{w} - \tilde{w}_0)}{\lambda}. \end{aligned} \tag{9.44}$$

Integrating both sides from $-T$ to 0, we get

$$\begin{aligned} & \int_{-T}^{0} \tilde{w}^* W^{-1} \tilde{w}\, d\tau - \lambda \int_{-T}^{0} \tilde{z}_1^* R^{-1} \tilde{z}_1\, d\tau = \\ & \qquad \int_{-T}^{0} (\tilde{w} - \tilde{w}_0)^* \Phi^{-1} (\tilde{w} - \tilde{w}_0)\, d\tau + \lambda \tilde{\epsilon}^*(-T) Y(-T) \tilde{\epsilon}(-T). \end{aligned} \tag{9.45}$$

Let $\epsilon > 0$ be sufficiently small. Since the map $\tilde{w} \to \tilde{w} - \tilde{w}_0$ and its inverse are bounded, there exists $\delta > 0$ such that

$$\int_{-T}^{0} (\tilde{w} - \tilde{w}_0)^* \Phi^{-1} (\tilde{w} - \tilde{w}_0)\, d\tau > (\delta + \lambda\epsilon) \int_{-T}^{0} \tilde{w}^* W^{-1} \tilde{w}\, d\tau. \tag{9.46}$$

Now by the assumption on $\tilde{\epsilon}^*(-T)Y(-T)\tilde{\epsilon}(-T)$, equation (9.43) follows, if $T \geq \tilde{T}$. □

We now go back to the original problem and reverse time in (9.40).

THEOREM 9.1. *On $[0, T]$, let*

$$\begin{aligned} \dot{Y} = {} & (\tilde{A} - \tilde{W}_2 S_1 - \tilde{W}_4 T_1) Y + Y(\tilde{A}^* + \tilde{C}^* S_2 + \tilde{B}^* D^* T_2) \\ & + Y(\tilde{C}^* S_1 + \tilde{B}^* D^* T_1) Y - (\tilde{W}_1 + \tilde{W}_2 S_2 + \tilde{W}_4 T_2), \quad Y(0) = 0. \end{aligned} \tag{9.47}$$

$$L = (S_1 Y + S_2)^*. \tag{9.48}$$

Consider (9.12). Let $\tilde{T}$ be as defined in Lemma 9.2. If $T \geq \tilde{T}$, there exists $\tilde{\delta} > 0$ such that

$$\int_0^T v^* R v \, dt - \lambda \int_0^T z^* W z \, dt \geq \tilde{\delta} \int_0^T v^* R v \, dt. \tag{9.49}$$

Proof. From equation (9.43) of the dual system, we deduce that if $T \geq \tilde{T}$,

$$\int_0^T w^* R w \, dt - \lambda \int_0^T z_1^* W z_1 \, dt \geq \delta_1 \int_0^T w^* R w \, dt, \tag{9.50}$$

for some $\delta_1 > 0$. Since the map $w \to v$ is bounded, there exists $\tilde{\delta} > 0$ such that

$$\int_0^T w^* R w \, dt - \lambda \int_0^T z_1^* W z_1 \, dt \geq \tilde{\delta} \int_0^T v^* R v \, dt. \tag{9.51}$$

From (9.13), (9.23), (9.24), and (9.51), we get (9.49). □

The above theorem shows that for the controller defined by (9.5) and (9.6), the performance is greater than λ. In fact from equation (9.49), we have

$$\inf_{v \neq 0} \frac{\int_0^T v^* R v \, dt}{\int_0^T z^* W z \, dt} > \lambda. \tag{9.52}$$

10. Summary of Results

The system is given by

$$\dot{x} = A(t)x + B_1(t)u + B_2(t)v, \quad x(0) = 0, \tag{10.1}$$

$$z = C(t)x + D(t)u + E(t)v, \tag{10.2}$$

$$y = C_2(t)x + D_2(t)u + E_2(t)v. \tag{10.3}$$

Let

$$\lambda_{opt} = \max_u \min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v \, dt}{\int_0^T \frac{1}{2} z^* W z \, dt}, \tag{10.4}$$

and let $\lambda < \lambda_{opt}$. Also, let $W_1, \cdots, W_6$ be defined by

$$z^* W z = x^* W_1 x + 2x^* W_2 u + u^* W_3 u + 2x^* W_4 v + v^* W_5 v + 2u^* W_6 v. \tag{10.5}$$

The relevant controller equations are

$$\begin{aligned}
\Omega &= (R - \lambda W_5)^{-1}, && (10.6)\\
\Lambda &= \{W_3 + \lambda W_6 \Omega W_6^*\}^{-1}, && (10.7)\\
U_1 &= \Lambda(B_1^* + \lambda W_6 \Omega B_2^*), && (10.8)\\
U_2 &= -\Lambda(W_2^* + \lambda W_6 \Omega W_4^*), && (10.9)\\
V_1 &= \lambda\Omega(-B_2^* + W_6^* U_1), && (10.10)\\
V_2 &= \lambda\Omega(W_4^* + W_6^* U_2), && (10.11)
\end{aligned}$$

$$\begin{aligned}
&\dot{P} + P(A + B_1 U_2 + B_2 V_2) + (A^* - W_2 U_1 - W_4 V_1)P \\
&\quad + P(B_1 U_1 + B_2 V_1)P - (W_1 + W_2 U_2 + W_4 V_2) = 0, \quad P(T) = 0. \qquad (10.12)
\end{aligned}$$

Note that the above equation is symmetric.

Let

$$\tilde{z}_1 = B_2^* \tilde{\epsilon} + E_2^* \tilde{u} + E^* \tilde{w}, \qquad (10.13)$$

and let $\tilde{W}_1, \cdots, \tilde{W}_6$ be defined by

$$\tilde{z}_1^* R^{-1} \tilde{z}_1 = \tilde{\epsilon}^* \tilde{W}_1 \tilde{\epsilon} + 2\tilde{\epsilon}^* \tilde{W}_2 \tilde{u} + \tilde{u}^* \tilde{W}_3 \tilde{u} + 2\tilde{\epsilon}^* \tilde{W}_4 \tilde{w} + \tilde{w}^* \tilde{W}_5 \tilde{w} + 2\tilde{u}^* \tilde{W}_6 \tilde{w}. \qquad (10.14)$$

The relevant observer equations are

$$\begin{aligned}
\Phi &= (W^{-1} - \lambda \tilde{W}_5)^{-1}, && (10.15)\\
\Gamma &= \{\tilde{W}_3 + \lambda \tilde{W}_6 \Phi \tilde{W}_6^*\}^{-1}, && (10.16)\\
\tilde{A} &= A + B_2 V_1 P + B_2 V_2, && (10.17)\\
\tilde{B} &= -(U_1 P + U_2), && (10.18)\\
\tilde{C} &= C_2 + E_2 V_1 P + E_2 V_2, && (10.19)\\
S_1 &= \Gamma(\tilde{C} + \lambda \tilde{W}_6 \Phi D \tilde{B}), && (10.20)\\
S_2 &= -\Gamma(\tilde{W}_2^* + \lambda \tilde{W}_6 \Phi \tilde{W}_4^*), && (10.21)\\
T_1 &= \lambda\Phi(-D\tilde{B} + \tilde{W}_6^* S_1), && (10.22)\\
T_2 &= \lambda\Phi(\tilde{W}_4^* + \tilde{W}_6^* S_2). && (10.23)
\end{aligned}$$

$$\begin{aligned}
&\dot{Y} = (\tilde{A} - \tilde{W}_2 S_1 - \tilde{W}_4 T_1)Y + Y(\tilde{A}^* + \tilde{C}^* S_2 + \tilde{B}^* D^* T_2) \\
&\quad + Y(\tilde{C}^* S_1 + \tilde{B}^* D^* T_1)Y - (\tilde{W}_1 + \tilde{W}_2 S_2 + \tilde{W}_4 T_2), \quad Y(0) = 0. \qquad (10.24)
\end{aligned}$$

Note that the above equation is symmetric.

The suboptimal controller is given by

$$\dot{q} = Aq + B_1(U_1Pq + U_2q) + B_2(V_1Pq + V_2q) + L(\tilde{C}q + D_2u - y), \tag{10.25}$$
$$L = (S_1Y + S_2)^*, \tag{10.26}$$
$$u = (U_1P + U_2)q. \tag{10.27}$$

11. CONCLUSIONS

In this report we treated the suboptimal finite horizon H_∞ control problem. A differential equation for the measure of performance is derived. The report also illustrates the usefulness of the finite horizon techniques in computing the infimal H_∞ norm in the infinite horizon case. An expression for a suboptimal finite horizon H_∞ controller is derived in a generalized case. The general case has been treated directly without the utilization of transformations. The ouput feedback controller is synthesized via a controller component and an observer component. A summary of all the design equations is given and these equations are easy to program on a digital computer. In the time-invariant case the dynamic Riccati equations involved in the design usually converge to constant matrices. The theory developed in this report is useful for the worst case design of flight control systems of aircraft in the presence of disturbances, commands, and sensor noise. The controller equations are in a form that is convenient for the design of flight control systems of advanced aircraft, since the results are applicable for a very general case. Preliminary applications to flight control design have shown considerable promise and the next phase of research will be directed towards the development of a systematic methodology for the design of a flight control system for an advanced aircraft. Further research also needs to be done to assess the relationship of the weighting matrices to satisfactory performance.

12. FUTURE WORK

Software is already in place to predict the best aircraft performance under worst case conditions. The success of this portion of the software is demonstrated in Section 6 of this report by way of examples. Future work based on the theoretical results of the report would entail the following:

1. Making use of the theory developed in Sections 7-9, software will be developed to synthesize both state and output feedback controllers.
2. Making use of an advanced aircraft model during landing, flight control laws will be developed to achieve satisfactory performance in the presence of disturbances, commands, and variations of the model.
3. The performance of the aircraft will be compared with that achieved through other techniques and the benefits, if any, will be demonstrated.

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